

# Tomita-Takesaki Modular Theory vs. Quantum Information Theory

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## Abstract

In this review article, we make an attempt to find out the relationship between separating and cyclic vectors in the theory of von Neumann algebra and entangled states in the theory of quantum information. The corresponding physical interpretation is presented as well.

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# 1 von Neumann algebras

Let  $\mathcal{H}$  be a Hilbert space for which  $L(\mathcal{H})$  stands for the all bounded linear operator acting on it. Assume that  $\mathcal{M}$  is a subset of  $L(\mathcal{H})$ , i.e.,  $\mathcal{M} \subseteq L(\mathcal{H})$ . We denote by  $\mathcal{M}'$  its commutant, i.e., the set of all bounded operators on  $\mathcal{H}$  commuting with every operator in  $\mathcal{M}$ :

$$\mathcal{M}' \stackrel{\text{def}}{=} \{M' \in L(\mathcal{H}) : [M', M] = 0 \text{ for all } M \in \mathcal{M}\}.$$

Clearly,  $\mathcal{M}'$  is a Banach algebra of operators containing the identity  $\mathbb{1}$ . If  $\mathcal{M}$  is self-adjoint, then  $\mathcal{M}'$  is a  $C^*$ -algebra of operators on  $\mathcal{H}$ . One has

$$\begin{aligned} \mathcal{M} &\subseteq \mathcal{M}'' = \mathcal{M}^{(\text{iv})} = \mathcal{M}^{(\text{vi})} = \dots \\ \mathcal{M}' &= \mathcal{M}''' = \mathcal{M}^{(\text{v})} = \mathcal{M}^{(\text{vii})} = \dots \end{aligned}$$

**Definition 1.1.** A von Neumann algebra on  $\mathcal{H}$  is a  $*$ -algebra  $\mathcal{M}$  of  $L(\mathcal{H})$  such that

$$\mathcal{M} = \mathcal{M}''.$$

The *center* of a von Neumann algebra  $\mathcal{M}$  is defined by

$$\mathcal{C} \stackrel{\text{def}}{=} \mathcal{M} \cap \mathcal{M}'.$$

A von Neumann algebra is called a *factor* if it has a trivial center, i.e., if  $\mathcal{C} = \mathbb{C}\mathbb{1}$ .

**Definition 1.2.** If  $\mathcal{M}$  is a subset of  $L(\mathcal{H})$  and  $\mathcal{H}_0$  is a subset of  $\mathcal{H}$ , let  $[\mathcal{M}\mathcal{H}_0]$  denote the closure of the linear span of elements of the form  $M|u\rangle$ , where  $M \in \mathcal{M}, |u\rangle \in \mathcal{H}_0$ . Let  $[\mathcal{M}\mathcal{H}_0]$  also denote the orthogonal projection onto  $[\mathcal{M}\mathcal{H}_0]$ .

**Proposition 1.3.** Let  $\text{Tr}$  be the usual trace on  $L(\mathcal{H})$ , and let  $\mathcal{C}_1$  be the Banach space of trace-class operators on  $\mathcal{H}$  equipped with the trace norm  $T \mapsto \text{Tr}(|T|) \stackrel{\text{def}}{=} \|T\|_1$ . Then it follows that  $L(\mathcal{H})$  is the dual  $\mathcal{C}_1^*$  of  $\mathcal{C}_1$  by the duality:

$$(A, T) \in L(\mathcal{H}) \times \mathcal{C}_1 \longmapsto \text{Tr}(AT).$$

**Definition 1.4.** The space of  $\sigma$ -weakly continuous linear functionals on  $L(\mathcal{H})$  is called the *predual* of  $L(\mathcal{H})$  and is denoted by  $L(\mathcal{H})_*$ .

As noted in Proposition 1.3,  $L(\mathcal{H})_*$  can be canonically identified with  $\mathcal{C}_1$  and  $L(\mathcal{H}) = (L(\mathcal{H})_*)^*$ .

## 1.1 Normal states and the predual

If  $\mu$  is a  $\sigma$ -finite measure, the  $L^\infty(d\mu)$  forms a von Neumann algebra of multiplication operators on the Hilbert space  $L^2(d\mu)$ .  $L^\infty(d\mu)$  is the dual of  $L^1(d\mu)$ ;  $L^1(d\mu)$ , however, is only a norm-closed subspace of the dual of  $L^\infty(d\mu)$ .

In this section, we single out an analogous subset of the dual of a von Neumann algebra  $\mathcal{M}$ , called the *predual*, and study its properties.

**Definition 1.5.** The *predual* of a von Neumann algebra  $\mathcal{M}$  is the space of all  $\sigma$ -weakly continuous functionals on  $\mathcal{M}$ . It is denoted by  $\mathcal{M}_*$ .

Note that all elements of  $\omega \in \mathcal{M}_*$  have the form

$$\omega(M) = \sum_n \langle \xi_n | M | \eta_n \rangle,$$

where  $\sum_n \|\xi_n\|^2 < +\infty$  and  $\sum_n \|\eta_n\|^2 < +\infty$ .

**Remark 1.6.** Now we remark here that we can construct an operator (in Dirac notation)

$$\sum_n |\xi_n\rangle\langle\eta_n|$$

when  $\sum_n \|\xi_n\|^2 < +\infty$  and  $\sum_n \|\eta_n\|^2 < +\infty$ .

Since

$$\begin{aligned} \left\| \sum_n |\xi_n\rangle\langle\eta_n| \right\| &\leq \sum_n \|\xi_n\rangle\langle\eta_n|\| = \sum_n \|\xi_n\rangle\| \|\eta_n\rangle\| \\ &\leq \left( \sum_n \|\xi_n\rangle\|^2 \right) \left( \sum_n \|\eta_n\rangle\|^2 \right) < +\infty, \end{aligned}$$

it follows that  $\sum_n |\xi_n\rangle\langle\eta_n| \in L(\mathcal{H})$ . Therefore, each normal element  $\omega \in \mathcal{M}_*$  has a representative in  $L(\mathcal{H})$ :

$$\omega(M) = \left\langle \sum_n |\xi_n\rangle\langle\eta_n|, M \right\rangle_{\text{HS}},$$

where  $\langle X, Y \rangle_{\text{HS}} \stackrel{\text{def}}{=} \text{Tr}(X^*Y)$ .

**Proposition 1.7.** *The predual  $\mathcal{M}_*$  of a von Neumann algebra  $\mathcal{M}$  is a Banach space in the norm of  $\mathcal{M}^*$ , and  $\mathcal{M}$  is the dual of  $\mathcal{M}_*$  in the duality*

$$(M, \omega) \in \mathcal{M} \times \mathcal{M}_* \longmapsto \omega(M).$$

**Remark 1.8.** We recall the following identifications:

$$\ell_1^* = \ell_\infty, \quad L_1^* = L_\infty, \quad \mathcal{C}_1^* = L(\mathcal{H}).$$

Thus the predual  $\mathcal{M}_*$  of von Neumann algebra  $\mathcal{M}$  can be viewed as an analog of  $\mathcal{C}_p$ -class with  $p = 1$  in  $L(\mathcal{H})$ . Therefore, we have that if denote  $\mathcal{M}_\infty \equiv \mathcal{M}$  and  $\mathcal{M}_1 \equiv \mathcal{M}_*$

$$(\mathcal{M}_*)^* = \mathcal{M}, \quad \text{or} \quad \mathcal{M}_1^* = \mathcal{M}_\infty.$$

In particular, when  $\mathcal{M} = L(\mathcal{H})$ , we have  $L(\mathcal{H})_1^* = L(\mathcal{H})_\infty$ .

**Proposition 1.9.** *Let  $\omega$  be a state on a von Neumann algebra  $\mathcal{M}$  acting on a Hilbert space  $\mathcal{H}$ . Then the following conditions are equivalent:*

- (i)  $\omega$  is normal;
- (ii)  $\omega$  is  $\sigma$ -weakly continuous;
- (iii) there exist a density matrix—a positive trace-class operator  $D_\omega$  on  $\mathcal{H}$  with  $\text{Tr}(D_\omega) = 1$ —such that  $\omega(M) = \text{Tr}(D_\omega M)$  for all  $M$  in  $\mathcal{M}$ .

**Remark 1.10.** We recall that Riesz Representation Theorem describes continuous functional on a Hilbert space has a vector representative: If  $f$  is a continuous functional on a Hilbert space  $\mathcal{H}$ , then there is a vector  $|u_f\rangle \in \mathcal{H}$  such that

$$f(|v\rangle) = \langle u_f | v \rangle, \quad \forall |v\rangle \in \mathcal{H}.$$

By comparison with Riesz Representation Theorem, we have: For each normal state  $\omega \in \mathcal{M}_*$ , it has a representative  $D_\omega$  in  $L(\mathcal{H})$  as follows:

$$\omega(M) = \langle D_\omega, M \rangle_{\text{HS}}.$$

By the definition of normal element in  $\mathcal{M}_*$ , there exist a sequence of vectors  $\{|\psi_n\rangle\}$  in  $\mathcal{H}$ ,  $\sum_n \|\psi_n\|^2 < +\infty$ , such that

$$D_\omega = \sum_n |\psi_n\rangle\langle\psi_n|.$$

Furthermore, setting  $\lambda_n \stackrel{\text{def}}{=} \|\psi_n\|^2 > 0$  and  $|\psi_n\rangle = \sqrt{\lambda_n}|\phi_n\rangle$  with  $\|\phi_n\| = 1$ , we have

$$D_\omega = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n|.$$

**Proposition 1.11.** *Let  $\mathcal{M}$  be a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$ . If  $\mathcal{I}$  be a  $\sigma$ -weakly closed two-sided ideal in  $\mathcal{M}$ . Then there exists a projection  $E \in \mathcal{M} \cap \mathcal{M}'$  such that  $\mathcal{I} = E\mathcal{M}E$ .*

## 1.2 $\sigma$ -finite von Neumann algebras

**Definition 1.12.** A von Neumann algebra  $\mathcal{M}$ , acting on a Hilbert space  $\mathcal{H}$ , is  $\sigma$ -finite if all collections of mutually orthogonal projections have at most a countable cardinality.

**Definition 1.13.** Let  $\mathcal{M}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$ . A subset  $\mathcal{H}_0 \subseteq \mathcal{H}$  is *cyclic* for  $\mathcal{M}$  if the set  $\{M|u\rangle : M \in \mathcal{M}, |u\rangle \in \mathcal{H}_0\}$  is dense in  $\mathcal{H}$ , i.e.,  $[\mathcal{M}\mathcal{H}_0] = \mathcal{H}$ . We say that  $\mathcal{H}_0$  is *separating* for  $\mathcal{M}$  if for any  $M \in \mathcal{M}$ ,  $M|u\rangle = 0$  for all  $|u\rangle \in \mathcal{H}_0$  implies  $M = 0$ .

**Proposition 1.14.** *Let  $\mathcal{M}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$  and  $\mathcal{H}_0 \subseteq \mathcal{H}$  a subset. Then  $\mathcal{H}_0$  is cyclic for  $\mathcal{M}$  if and only if  $\mathcal{H}_0$  is separating for  $\mathcal{M}'$ .*

**Definition 1.15.** Let  $\mathcal{M}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$ . A vector  $|\Omega\rangle$  is called *cyclic* for  $\mathcal{M}$  if the set  $\{M|\Omega\rangle : M \in \mathcal{M}\}$  is dense in  $\mathcal{H}$ , i.e.,  $[\mathcal{M}|\Omega\rangle] = \mathcal{H}$ . We say that  $|\Omega\rangle \in \mathcal{H}$  is *separating* for  $\mathcal{M}$  if for any  $M \in \mathcal{M}$ ,  $M|\Omega\rangle = 0$  implies  $M = 0$ .

**Proposition 1.16.** *Let  $\mathcal{M}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$  and  $|\Omega\rangle \in \mathcal{H}$ . Then  $|\Omega\rangle$  is cyclic for  $\mathcal{M}$  if and only if  $|\Omega\rangle$  is separating for  $\mathcal{M}'$ .*

**Definition 1.17.** A state  $\omega$  on a von Neumann algebra  $\mathcal{M}$  is *faithful* if  $\omega(M) > 0$  for all nonzero  $M \in \mathcal{M}^+$ .

**Proposition 1.18.** *Let  $\mathcal{M}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$ . The the following four conditions are equivalent:*

- (i)  $\mathcal{M}$  is  $\sigma$ -finite;
- (ii) there exists a countable subset of  $\mathcal{H}$  which is separating for  $\mathcal{M}$ ;
- (iii) there exists a faithful normal state on  $\mathcal{M}$ ;
- (iv)  $\mathcal{M}$  is isomorphic with a von Neumann algebra  $\pi(\mathcal{M})$  which admits a separating and cyclic vector.

### 1.3 Tomita-Takesaki modular theory

*Tomita-Takesaki Modular Theory* has been one of the most exciting subjects for operator algebras and for its applications to mathematical physics. We will give here a short introduction to this theory and state some of its main results.

If von Neumann algebra  $\mathcal{M}$  is a  $\sigma$ -finite, we may assume that  $\mathcal{M}$  has a separating and cyclic vector  $|\Omega\rangle$ . In Tomita-Takesaki modular theory, one studies systematically the relation of a von Neumann algebra  $\mathcal{M}$  and its commutant  $\mathcal{M}'$  in the case where both algebras have a common cyclic vector  $|\Omega\rangle$ . The mapping

$$M \in \mathcal{M} \longmapsto M|\Omega\rangle \in \mathcal{H},$$

then establishes a one-to-one linear correspondence between  $\mathcal{M}$  and a dense subspace  $\mathcal{M}|\Omega\rangle$  of  $\mathcal{H}$ . This correspondence may be used to transfer algebraic operations on  $\mathcal{M}$  to operations on  $\mathcal{M}|\Omega\rangle$ .

The two anti-linear operators  $S_0$  and  $F_0$ , given by

$$\begin{aligned} S_0 M|\Omega\rangle &= M^*|\Omega\rangle, \quad \forall M \in \mathcal{M}, \\ F_0 M'|\Omega\rangle &= M'^*|\Omega\rangle, \quad \forall M' \in \mathcal{M}', \end{aligned}$$

are both well-defined on the dense domains  $D(S_0) = \mathcal{M}|\Omega\rangle$  and  $D(F_0) = \mathcal{M}'|\Omega\rangle$ .

**Proposition 1.19.**  *$S_0$  and  $F_0$  are closable. And*

$$S_0^* = \overline{F_0}, \quad F_0^* = \overline{S_0},$$

where the bar denotes the closure.

**Definition 1.20.** Define  $S$  and  $F$  as the closures of  $S_0$  and  $F_0$ , respectively, i.e.,

$$S = \overline{S_0}, \quad F = \overline{F_0}.$$

Let  $\Delta$  be the unique, positive, self-adjoint operator and  $J$  the unique anti-unitary operator occurring in the *polar decomposition*

$$S = J\Delta^{\frac{1}{2}}$$

of  $S$ .  $\Delta$  is called the *modular operator associated with the pair*  $\{\mathcal{M}, |\Omega\rangle\}$  and  $J$  is called the *modular conjugation*.

**Proposition 1.21.** *The following relations are valid:*

$$\left\{ \begin{array}{l} \Delta = FS \\ \Delta^{-1} = SF \end{array} \right\}, \quad \left\{ \begin{array}{l} S = J\Delta^{\frac{1}{2}} \\ F = J\Delta^{-\frac{1}{2}} \end{array} \right\}, \quad \left\{ \begin{array}{l} J^* = J \\ J^2 = \mathbb{1} \end{array} \right\}, \quad \Delta^{-\frac{1}{2}} = J\Delta^{\frac{1}{2}}J.$$

**Theorem 1.22** (Tomita-Takesaki Theorem). *Let  $\mathcal{M}$  be a von Neumann algebra with cyclic and separating vector  $|\Omega\rangle$ , and let  $\Delta$  be the associated modular operator and  $J$  the associated modular conjugation. It follows that*

$$\left\{ \begin{array}{l} J\mathcal{M}J = \mathcal{M}', \\ \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, \forall t \in \mathbb{R}. \end{array} \right.$$

## 1.4 Self-dual cones and standard forms

**Definition 1.23.** The *natural positive cone*  $\mathcal{P}$  associated with the pair  $(\mathcal{M}, |\Omega\rangle)$  is defined as the closure of the set

$$\{Mj(M)|\Omega\rangle : M \in \mathcal{M}\},$$

where  $j : \mathcal{M} \mapsto \mathcal{M}'$  is the anti-linear  $*$ -isomorphism defined by

$$j(M) \stackrel{\text{def}}{=} JMJ, \quad \forall M \in \mathcal{M}.$$

**Proposition 1.24.** *The closed subset  $\mathcal{P} \subseteq \mathcal{H}$  has the following properties:*

(i)

$$\begin{aligned} \mathcal{P} &= [\Delta^{\frac{1}{4}}\mathcal{M}^+|\Omega\rangle] = [\Delta^{-\frac{1}{4}}\mathcal{M}'^+|\Omega\rangle] \\ &= [\Delta^{\frac{1}{4}}[\mathcal{M}^+|\Omega\rangle]] = [\Delta^{-\frac{1}{4}}[\mathcal{M}'^+|\Omega\rangle]] \end{aligned}$$

and hence  $\mathcal{P}$  is a convex cone;

- (ii)  $\Delta^{it}\mathcal{P} = \mathcal{P}$  for all  $t \in \mathbb{R}$ ;
- (iii) if  $f$  is a positive-definite function, then  $f(\log \Delta)\mathcal{P} \subseteq \mathcal{P}$ ;
- (iv) if  $|\xi\rangle \in \mathcal{P}$ , then  $J|\xi\rangle = |\xi\rangle$ ;
- (v) if  $M \in \mathcal{M}$ , then  $Mj(M)\mathcal{P} \subseteq \mathcal{P}$ .

**Proposition 1.25.** (i)  $\mathcal{P}$  is a self-adjoint cone, i.e.,  $\mathcal{P} = \mathcal{P}^\vee$ , where

$$\mathcal{P}^\vee = \{|\eta\rangle \in \mathcal{H} : \langle \xi | \eta \rangle \geq 0 \text{ for all } |\xi\rangle \in \mathcal{P}\}.$$

- (ii)  $\mathcal{P}$  is a pointed cone, i.e.,

$$\mathcal{P} \cap (-\mathcal{P}) = \{0\}.$$

- (iii) If  $J|\xi\rangle = |\xi\rangle$ , then  $|\xi\rangle$  has a unique decomposition  $|\xi\rangle = |\xi_1\rangle - |\xi_2\rangle$ , where  $|\xi_1\rangle, |\xi_2\rangle \in \mathcal{P}$  and  $|\xi_1\rangle \perp |\xi_2\rangle$ .
- (iv)  $\mathcal{H}$  is linearly spanned by  $\mathcal{P}$ .

**Proposition 1.26** (Universality of the cone  $\mathcal{P}$ ). (i) If  $|\xi\rangle \in \mathcal{P}$ , then  $|\xi\rangle$  is cyclic for  $\mathcal{M}$  if and only if  $|\xi\rangle$  is separating for  $\mathcal{M}$ .

- (ii) If  $|\xi\rangle \in \mathcal{P}$  is cyclic and separating, then the modular conjugation  $J_{|\xi\rangle}$  and the natural positive cone  $\mathcal{P}_{|\xi\rangle}$  associated with the pair  $(\mathcal{M}, |\xi\rangle)$  satisfy

$$J_{|\xi\rangle} = J, \quad \mathcal{P}_{|\xi\rangle} = \mathcal{P}.$$

**Theorem 1.27** (Araki). For each  $|\xi\rangle \in \mathcal{P}$ , define the normal positive form  $\omega_{|\xi\rangle} \in \mathcal{M}_{*,+}$  by

$$\omega_{|\xi\rangle}(M) = \langle \xi | M | \xi \rangle, \quad M \in \mathcal{M}.$$

It follows that

- (i) for any  $\omega \in \mathcal{M}_{*,+}$ , there exists a unique  $|\xi\rangle \in \mathcal{P}$  such that  $\omega = \omega_{|\xi\rangle}$ ,
- (ii) the mapping  $|\xi\rangle \mapsto \omega_{|\xi\rangle}$  is a homeomorphism when both  $\mathcal{P}$  and  $\mathcal{M}_{*,+}$  are equipped with the norm topology. Moreover, the following estimates are valid:

$$\| |\xi\rangle - |\eta\rangle \|^2 \leq \| \omega_{|\xi\rangle} - \omega_{|\eta\rangle} \| \leq \| |\xi\rangle - |\eta\rangle \| \| |\xi\rangle + |\eta\rangle \|.$$



## 2 The operator-vector correspondence

For the operator-vector correspondence, we distinguish two situations where slight differences occurred in the corresponding definitions.

### 2.1 vec mapping in unipartite operator spaces

It will be helpful throughout this course to make use of a simple correspondence between the spaces  $L(\mathcal{X}, \mathcal{Y})$  and  $\mathcal{Y} \otimes \mathcal{X}$ , for given complex Euclidean spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . We define the mapping

$$\text{vec} : L(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathcal{Y} \otimes \mathcal{X}$$

to be the linear mapping that represents a change of bases from the standard basis of  $L(\mathcal{X}, \mathcal{Y})$  to the standard basis of  $\mathcal{Y} \otimes \mathcal{X}$ . Specifically, we define

$$\text{vec}(E_{\mu, \nu}) = e_{\mu} \otimes e_{\nu}$$

for all  $\mu \in \Sigma$  and  $\nu \in \Gamma$ , at which point the mapping is determined for every  $A \in L(\mathcal{X}, \mathcal{Y})$  by linearity. In the Dirac notation, this mapping amounts to flipping a bra to a ket:

$$\text{vec}(|\mu\rangle\langle\nu|) = |\mu\rangle \otimes |\nu\rangle \equiv |\mu\rangle|\nu\rangle \equiv |\mu\nu\rangle.$$

(Note that it is only standard basis elements that are flipped in this way.)

The vec mapping is a linear bijection, which implies that every vector  $|u\rangle \in \mathcal{Y} \otimes \mathcal{X}$  uniquely determines an operator  $A \in L(\mathcal{X}, \mathcal{Y})$  that satisfies  $\text{vec}(A) = |u\rangle$ . It is also an isometry, in the sense that

$$\langle A, B \rangle = \langle \text{vec}(A), \text{vec}(B) \rangle$$

for all  $A, B \in L(\mathcal{X}, \mathcal{Y})$ . The following properties of the vec mapping are easily verified:

- (i) For every choice of complex Euclidean spaces  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ , and  $\mathcal{Y}_2$ , and every choice of operators  $A \in L(\mathcal{X}_1, \mathcal{Y}_1)$ ,  $B \in L(\mathcal{X}_2, \mathcal{Y}_2)$ , and  $X \in L(\mathcal{X}_2, \mathcal{X}_1)$ , it holds that

$$(A \otimes B) \text{vec}(X) = \text{vec}(AXB^{\top}). \quad (2.1)$$

- (ii) For every choice of complex Euclidean spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , and every choice of operators  $A, B \in L(\mathcal{X}, \mathcal{Y})$ , the following equations hold:

$$\text{Tr}_{\mathcal{X}}(\text{vec}(A) \text{vec}(B)^*) = AB^*, \quad (2.2)$$

$$\text{Tr}_{\mathcal{Y}} (\text{vec}(A) \text{vec}(B)^*) = (B^* A)^\top. \quad (2.3)$$

(iii) For  $|u\rangle \in \mathcal{X}$  and  $|v\rangle \in \mathcal{Y}$  we have

$$\text{vec}(|u\rangle\langle v|) = |u\rangle \otimes \overline{|v\rangle}. \quad (2.4)$$

This includes the special cases  $\text{vec}(|u\rangle) = |u\rangle$  and  $\text{vec}(\langle v|) = \overline{|v\rangle}$ .

**Example 2.1** (The Schmidt decomposition). Suppose  $|u\rangle \in \mathcal{Y} \otimes \mathcal{X}$  for given complex Euclidean spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . Let  $A \in \text{L}(\mathcal{X}, \mathcal{Y})$  be the unique operator for which  $|u\rangle = \text{vec}(A)$ . There exists a singular value decomposition

$$A = \sum_{i=1}^r s_i |y_i\rangle\langle x_i|$$

of  $A$ . Consequently

$$|u\rangle = \text{vec}(A) = \text{vec}\left(\sum_{i=1}^r s_i |y_i\rangle\langle x_i|\right) = \sum_{i=1}^r s_i \text{vec}(|y_i\rangle\langle x_i|) = \sum_{i=1}^r s_i |y_i\rangle \otimes \overline{|x_i\rangle}.$$

The fact that  $\{|x_1\rangle, \dots, |x_r\rangle\}$  is orthonormal implies that  $\{\overline{|x_1\rangle}, \dots, \overline{|x_r\rangle}\}$  is orthonormal as well.

We have therefore established the validity of the Schmidt decomposition, which states that every vector  $|u\rangle \in \mathcal{Y} \otimes \mathcal{X}$  can be expressed in the form

$$|u\rangle = \sum_{i=1}^r s_i |y_i\rangle \otimes |z_i\rangle$$

for positive real numbers  $s_1, \dots, s_r$  and orthonormal sets

$$\{|y_1\rangle, \dots, |y_r\rangle\} \subset \mathcal{Y} \quad \text{and} \quad \{|z_1\rangle, \dots, |z_r\rangle\} \subset \mathcal{X}.$$

## 2.2 vec mapping in multipartite operator spaces

When the vec mapping is generalized to multipartite spaces, caution should be given to the bipartite case (multipartite situation similarly). Specifically, for given complex Euclidean spaces  $\mathcal{X}_{A/B}$  and  $\mathcal{Y}_{A/B}$ ,

$$\text{vec} : \text{L}(\mathcal{X}_A \otimes \mathcal{X}_B, \mathcal{Y}_A \otimes \mathcal{Y}_B) \longrightarrow \mathcal{Y}_A \otimes \mathcal{X}_A \otimes \mathcal{Y}_B \otimes \mathcal{X}_B$$

is defined to be the linear mapping that represents a change of bases from the standard basis of  $L(\mathcal{X}_A \otimes \mathcal{X}_B, \mathcal{Y}_A \otimes \mathcal{Y}_B)$  to the standard basis of  $\mathcal{Y}_A \otimes \mathcal{X}_A \otimes \mathcal{Y}_B \otimes \mathcal{X}_B$ . Concretely,

$$\text{vec}(|m\rangle\langle n| \otimes |\mu\rangle\langle \nu|) := |mn\rangle \otimes |\mu\nu\rangle \equiv |mn\mu\nu\rangle,$$

where  $\{|n\rangle\}$  is an orthonormal basis for  $\mathcal{X}_A$  and  $\{|\nu\rangle\}$  is an orthonormal basis for  $\mathcal{X}_B$ , while  $\{|m\rangle\}$  is an orthonormal basis for  $\mathcal{Y}_A$  and  $\{|\mu\rangle\}$  is an orthonormal basis for  $\mathcal{Y}_B$ . Analogously, the mapping is determined for every operator  $X \in L(\mathcal{X}_A \otimes \mathcal{X}_B, \mathcal{Y}_A \otimes \mathcal{Y}_B)$  by linearity. Note that if  $X = A \otimes B$ , where  $A \in L(\mathcal{X}_A, \mathcal{Y}_A)$  and  $B \in L(\mathcal{X}_B, \mathcal{Y}_B)$ , then

$$\text{vec}(A \otimes B) = \text{vec}(A) \otimes \text{vec}(B).$$

### 3 Explicit examples

**Example 3.1.** Let  $\mathcal{H}_d$  be a  $d$ -dimensional complex Hilbert space. Consider a von Neumann algebra  $\mathcal{M} \equiv L(\mathcal{H}_d)$ . For any  $X \in L(\mathcal{H}_d)$ , the following map defined a faithful representation of von Neumann algebra  $\mathcal{M}$  on a Hilbert space  $\mathcal{H} \equiv \mathcal{H}_d \otimes \mathcal{H}_d$ :

$$\pi : X \longmapsto \pi(X) = X \otimes \mathbb{1}_d.$$

Setting  $|\Omega\rangle \stackrel{\text{def}}{=} \text{vec}(\mathbb{1}_d)$ , we have that  $|\Omega\rangle$  is a separating and cyclic vector in  $\mathcal{H}$  for von Neumann algebra  $\pi(\mathcal{M}) \equiv L(\mathcal{H}_d) \otimes \mathbb{1}_d$ . Therefore we can conclude that von Neumann algebra  $\mathcal{M}$  have a standard representation  $(\pi(\mathcal{M}), \mathcal{H}, |\Omega\rangle)$ .

Consider the Tomita-Takesaki modular theory in  $(\pi(\mathcal{M}), \mathcal{H}, |\Omega\rangle)$ . According to the Tomita-Takesaki modular theory

$$S\pi(X)|\Omega\rangle \stackrel{\text{def}}{=} \pi(X)^*|\Omega\rangle = (X^* \otimes \mathbb{1}_d)|\Omega\rangle, \quad \forall X \in \mathcal{M},$$

which is equivalently described as

$$S \text{vec}(X) = \text{vec}(X^*), \quad \forall X \in \mathcal{M}.$$

If we assume that  $K$  is the *complex conjugate operator* and  $P$  is a *swap operator*, then

$$\begin{aligned} S \text{vec}(X) &= \text{vec}(X^*) = \text{vec}((\overline{X})^\top) = P \text{vec}(\overline{X}) \\ &= PK \text{vec}(X) = KP \text{vec}(X), \end{aligned}$$

which means that  $S = PK = KP$ . Similarly,  $J = S = F = PK = KP$ , therefore  $\Delta = \mathbb{1}$ . In quantum physics,  $K$  stands for time reversal operation.

**Theorem 3.2.** *The set of all separating and cyclic vectors in  $\mathcal{H}$  for  $\pi(\mathcal{M})$  is precisely the set*

$$\{\text{vec}(A) \in \mathcal{H} : A \in \mathcal{M} \text{ is not singular}\}.$$

*Proof.* If  $A \in \mathcal{M}$  is not singular, then for any  $\pi(X) \in \pi(\mathcal{M})$ , we have

$$\pi(X) \text{vec}(A) = 0 \iff \text{vec}(XA) = 0 \iff XA = 0 \iff X = 0.$$

Thus  $\text{vec}(A)$  is a separating vector. When  $X$  is all over  $\mathcal{M}$ , we have

$$\pi(\mathcal{M}) \text{vec}(A) = \text{vec}(\mathcal{M}A) = \text{vec}(\mathcal{M}) = \mathcal{H},$$

which implies that  $\text{vec}(A)$  is a cyclic vector.

Now suppose that  $|\psi\rangle \in \mathcal{H}$  is a separating and cyclic vector for  $\pi(\mathcal{M})$ . Then there exists an operator  $B_\psi \in \mathcal{M}$  such that  $|\psi\rangle = \text{vec}(B_\psi)$ . If  $B_\psi$  is singular, then  $\mathcal{M}B_\psi$  is a proper left ideal of  $\mathcal{M}$ . Thus  $\pi(\mathcal{M})|\psi\rangle \neq \mathcal{H}$  and there exists  $X_1 \neq X_2$  such that  $X_1 B_\psi = X_2 B_\psi$ . That is,  $\pi(X_1)|\psi\rangle = \pi(X_2)|\psi\rangle$ . Therefore  $|\psi\rangle = \text{vec}(B_\psi)$  is not a separating and cyclic vector for singular operator  $B_\psi$ .  $\square$

**Remark 3.3.** We recall that the *Schmidt rank* of pure state  $|\psi\rangle \in \mathcal{H}$  is defined by

$$\text{SR}(|\psi\rangle) \stackrel{\text{def}}{=} \text{rank}(B_\psi), \quad |\psi\rangle = \text{vec}(B_\psi).$$

Hence the above result can be described equivalently as:

**Claim:**  $|\psi\rangle \in \mathcal{H}$  is a separating and cyclic vector for  $\pi(\mathcal{M})$  if and only if  $\text{SR}(|\psi\rangle) = d$ .

In some sense, separating and cyclic vectors stands for quantum states of most entanglement of measure.

If  $\omega$  is a state on  $\mathcal{M}$ , then there exist density matrix  $D_\omega \in \mathcal{L}(\mathcal{H}_d)$  such that

$$\omega(M) = \text{Tr}(D_\omega M) = \langle D_\omega, M \rangle_{\text{HS}}, \quad M \in \mathcal{L}(\mathcal{H}_d).$$

It is known that  $\omega$  is faithful if and only if  $D_\omega$  is not singular. Since  $\dim(\mathcal{H}_d) = d < +\infty$ , it follows that all states on  $\mathcal{M}$  are normal.

Consider the normalized vector  $|\Omega\rangle \stackrel{\text{def}}{=} \text{vec}(\sqrt{D_\omega}) \in \mathcal{H}$  for faithful normal state  $\omega$ . It is easily seen that

$$\omega(M) = \langle \Omega | \pi(M) | \Omega \rangle.$$

$|\Omega\rangle$  is a separating and cyclic vector  $\pi(\mathcal{M})$ . In terms of the language of quantum information theory,  $|\Omega\rangle$  is a purification of density matrix  $D_\omega$  in  $\mathcal{H}$ . Thus there is a connection between the standard representation of von Neumann algebra with a faithful normal state and *purification* of density matrix:

Given a faithful normal state  $\omega$  on von Neumann algebra  $\mathcal{M}$ . Then the standard representation of  $\mathcal{M}$  is  $(\pi(\mathcal{M}), \mathcal{H}, |\Omega\rangle)$ , where  $|\Omega\rangle = \text{vec}(\sqrt{D_\omega})$  is a purification of density matrix  $D_\omega$  which is not singular.

**Example 3.4** (Unification of finite or countable infinite situation). A simple example of the Tomita-Takesaki theory and its related KMS states can be built on the space of *Hilbert-Schmidt operators* on a Hilbert space. The set of Hilbert-Schmidt operators is itself a Hilbert space, and there are two preferred algebras of operators on it, which carry the modular structure.

Let  $\mathcal{H}$  be a (complex, separable) Hilbert space of dimension  $N$  (finite or infinite) and  $\{|\psi_i\rangle\}_{i=1}^N$  an orthonormal basis of it. We denote by  $\mathcal{C}_2$  the space of all Hilbert-Schmidt operators on  $\mathcal{H}$  ( $\mathcal{C}_2 \subset L(\mathcal{H})$ ). This is a Hilbert space with scalar product:  $(\mathcal{C}_2, \langle \cdot, \cdot \rangle_{\text{HS}})$

$$\langle X, Y \rangle_{\text{HS}} = \text{Tr}(X^* Y).$$

The vectors (an element of  $\mathcal{C}_2$  is called vector although it is operator on  $\mathcal{H}$ ),

$$\{E_{ij} = |\psi_i\rangle\langle\psi_j| : i, j = 1, 2, \dots, N\}$$

form an orthonormal basis of  $\mathcal{C}_2$ ,

$$\langle E_{ij}, E_{kl} \rangle = \delta_{ik} \delta_{jl}.$$

In particular, the vectors,

$$E_{ii} = |\psi_i\rangle\langle\psi_i|,$$

are one dimensional projection operators on  $\mathcal{H}$ . In what follows  $\mathbb{1}$  will denote the identity operator on  $\mathcal{H}$  and  $\mathbb{1}_2$  that on  $\mathcal{C}_2$  (in later notation:  $\mathbb{1}_2 = \mathbb{1} \boxtimes \mathbb{1}$ ).

All bounded linear operator acting on  $\mathcal{C}_2$  (i.e., linear super-operators in  $T(\mathcal{H})$ ) are denoted by  $L(\mathcal{C}_2)$ . We identify a special class of linear operators on  $\mathcal{C}_2$ , denoted by  $A \boxtimes B \in L(\mathcal{C}_2)$ ,  $A, B \in L(\mathcal{H})$ , which act on a vector  $X \in \mathcal{C}_2$  in the manner:

$$A \boxtimes B(X) \stackrel{\text{def}}{=} AXB^*.$$

Using the scalar product in  $\mathcal{C}_2$ , we see that

- (i)  $(A \boxtimes B)^* = A^* \boxtimes B^*$ ,
- (ii)  $(A_1 \boxtimes B_1)(A_2 \boxtimes B_2) = A_1 A_2 \boxtimes B_1 B_2$ .

Indeed,

$$\begin{aligned}
\langle (A \boxtimes B)^*(Y), X \rangle_{\mathbf{HS}} &= \langle Y, (A \boxtimes B)(X) \rangle_{\mathbf{HS}} = \text{Tr}(Y^* A X B^*) \\
&= \text{Tr}(B^* Y^* A X) = \langle (B^* Y^* A)^*, X \rangle_{\mathbf{HS}} \\
&= \langle A^* Y B, X \rangle_{\mathbf{HS}} = \langle A^* \boxtimes B^*(Y), X \rangle_{\mathbf{HS}},
\end{aligned}$$

which implies that  $(A \boxtimes B)^* = A^* \boxtimes B^*$ . Similar reasoning goes for  $(A_1 \boxtimes B_1)(A_2 \boxtimes B_2) = A_1 A_2 \boxtimes B_1 B_2$ .

There are two special von Neumann algebras which can be built out of these operators. These are,

$$\mathcal{A}_l \stackrel{\text{def}}{=} \{A_l = A \boxtimes \mathbb{1} : A \in L(\mathcal{H})\}, \quad \mathcal{A}_r \stackrel{\text{def}}{=} \{A_r = \mathbb{1} \boxtimes A : A \in L(\mathcal{H})\}.$$

As a matter of fact,  $A_l$  is a left regular representation of  $A$  or a left multiplication by  $A$ ;  $A_r$  is a right regular representation of  $A^*$  or a right multiplication by  $A^*$ . For any  $A, B \in L(\mathcal{H})$ , we have

$$A_l B_r = B_r A_l, \quad [A_l, B_r] = 0.$$

In fact, for any  $X \in \mathcal{C}_2$ ,

$$\begin{aligned}
A_l B_r(X) &= (A \boxtimes \mathbb{1})(\mathbb{1} \boxtimes B)(X) = (A \boxtimes \mathbb{1})(X B^*) \\
&= A X B^* = (\mathbb{1} \boxtimes B)(A X) = (\mathbb{1} \boxtimes B)(A \boxtimes \mathbb{1})(X) \\
&= B_r A_l(X).
\end{aligned}$$

They are mutual commutants and both are factors:

$$(\mathcal{A}_l)' = \mathcal{A}_r, \quad (\mathcal{A}_r)' = \mathcal{A}_l, \quad \mathcal{A}_l \cap \mathcal{A}_r = \mathbb{C}\mathbb{1}_2.$$

Consider now the operator  $J : \mathcal{C}_2 \longrightarrow \mathcal{C}_2$ , whose action on the vectors  $E_{ij}$  is given by

$$J E_{ij} \stackrel{\text{def}}{=} E_{ji} \implies J^2 = \mathbb{1}_2, \quad J(|\phi\rangle\langle\psi|) = |\psi\rangle\langle\phi|, \quad \forall |\phi\rangle, |\psi\rangle \in \mathcal{H}.$$

This operator is anti-unitary, and since

$$\begin{aligned}
[J(A \boxtimes \mathbb{1})J]E_{ij} &= J(A \boxtimes \mathbb{1})E_{ji} = J(A E_{ji}) \\
&= J(A|\psi_j\rangle\langle\psi_i|) = |\psi_i\rangle\langle\psi_j|A^* = (\mathbb{1} \boxtimes A)E_{ij},
\end{aligned}$$

we immediately get

$$J\mathcal{A}_l J = \mathcal{A}_r.$$

• **A KMS state.**

Let  $\{\lambda_i\}_{i=1}^N$  ( $N \leq +\infty$ ) be a sequence of non-zero, positive numbers, satisfying,  $\sum_{i=1}^N \lambda_i = 1$ . Let

$$\Omega \stackrel{\text{def}}{=} \sum_{i=1}^N \sqrt{\lambda_i} E_{ii} \in \mathcal{C}_2. \quad (3.1)$$

We note the following properties of  $\Omega$ .

- (i)  $\Omega$  defines a vector state  $\omega$  on the von Neumann algebra  $\mathcal{A}_l$ . This follows from the fact that for any  $A \boxtimes \mathbb{1} \in \mathcal{A}_l$ , we may define the state  $\omega$  on  $\mathcal{A}_l$  by

$$\omega(A \boxtimes \mathbb{1}) \stackrel{\text{def}}{=} \langle \Omega, (A \boxtimes \mathbb{1}) \Omega \rangle_{\text{HS}} = \text{Tr}(\Omega^* A \Omega) = \text{Tr}(D_\omega A), \quad D_\omega = \sum_{i=1}^N \lambda_i E_{ii}, \quad (3.2)$$

thus  $\Omega = D_\omega^{\frac{1}{2}}$ .

- (ii) The state  $\omega$  is faithful and normal. Normality follows from the last equality in Eq. (3.2) and the fact that  $D_\omega$  is a density matrix. To check for faithfulness, note that for any  $A \boxtimes \mathbb{1} \in \mathcal{A}_l$ ,

$$\omega((A \boxtimes \mathbb{1})^* (A \boxtimes \mathbb{1})) = \omega(A^* A \boxtimes \mathbb{1}) = \text{Tr}(D_\omega A^* A) = \sum_{i=1}^N \lambda_i \|A|\psi_i\rangle\|^2,$$

from which it follows that  $\omega((A \boxtimes \mathbb{1})^* (A \boxtimes \mathbb{1})) = 0$  if and only if  $A = 0$  (since the  $|\psi_i\rangle$  are an orthonormal basis set and the  $\lambda_i > 0$ ), hence if and only if  $A \boxtimes \mathbb{1} = 0$ .

- (iii) The vector  $\Omega$  is cyclic and separating for  $\mathcal{A}_l$ :  $[\mathcal{A}_l \Omega] = \mathcal{C}_2$ . Indeed, cyclicity follows from the fact that if  $X \in \mathcal{C}_2$  is orthogonal to all  $(A \boxtimes \mathbb{1})\Omega$ ,  $A \in \text{L}(\mathcal{H})$ , then

$$\langle X, (A \boxtimes \mathbb{1})\Omega \rangle_{\text{HS}} = \text{Tr}(X^* A \Omega) = \sum_{i=1}^N \sqrt{\lambda_i} \langle \psi_i | X^* A | \psi_i \rangle = 0, \quad \forall A \in \text{L}(\mathcal{H}).$$

Taking  $A = E_{kl}$ , we easily get from the above equality,  $\langle \psi_l | X^* | \psi_k \rangle = 0$  and since this holds for all  $k, l$ , we get  $X = 0$ . In the same way,  $\Omega$  is also cyclic for  $\mathcal{A}_r$ , hence separating for  $\mathcal{A}_l$ , i.e.,  $(A \boxtimes \mathbb{1})\Omega = (B \boxtimes \mathbb{1})\Omega \iff A \boxtimes \mathbb{1} = B \boxtimes \mathbb{1}$ .

We shall show in the sequel that the state  $\omega$  constructed above is indeed a KMS state for a particular choice of  $\lambda_i$ .

• **Time evolution and modular automorphism.**

We now construct a time evolution  $\sigma_t^\omega (t \in \mathbb{R})$ , on the algebra  $\mathcal{A}_l$ , using the state  $\omega$ , with respect to which it has the KMS property, for fixed  $\beta > 0$ ,

$$\omega(A_l \sigma_{t+i\beta}^\omega(B_l)) = \omega(\sigma_t^\omega(B_l) A_l), \quad \forall A_l, B_l \in \mathcal{A}_l,$$

and moreover the function,

$$F_{A_l, B_l}(z) \stackrel{\text{def}}{=} \omega(A_l \sigma_z^\omega(B_l)),$$

is analytic in the strip  $\{z \in \mathbb{C} : 0 < \text{Im}(z) < \beta\}$  and continuous on its boundaries. We start by defining the operators,

$$\mathbf{P}_{ij} \stackrel{\text{def}}{=} E_{ii} \boxtimes E_{jj}.$$

Clearly  $\mathbf{P}_{ij}$  are projection operators on the Hilbert space  $\mathcal{C}_2$ :

$$\begin{cases} \mathbf{P}_{ij}^* = \mathbf{P}_{ij}, \\ \mathbf{P}_{ij}^2 = \mathbf{P}_{ij}. \end{cases}$$

Indeed,

$$\begin{aligned} \mathbf{P}_{ij}^* &= (E_{ii} \boxtimes E_{jj})^* = E_{ii}^* \boxtimes E_{jj}^* = E_{ii} \boxtimes E_{jj} = \mathbf{P}_{ij}, \\ \mathbf{P}_{ij}^2 &= (E_{ii} \boxtimes E_{jj})^2 = E_{ii}^2 \boxtimes E_{jj}^2 = E_{ii} \boxtimes E_{jj} = \mathbf{P}_{ij}. \end{aligned}$$

Using  $D_\omega$  and for a fixed  $\beta > 0$ , define the operator  $H_\omega$  as:

$$D_\omega \stackrel{\text{def}}{=} e^{-\beta H_\omega} \implies H_\omega = -\frac{1}{\beta} \ln D_\omega = -\frac{1}{\beta} \sum_{i=1}^N (\ln \lambda_i) E_{ii}.$$

Clearly  $[D_\omega, H_\omega] = 0$ . Next we define the operators:

$$H_\omega^l \stackrel{\text{def}}{=} H_\omega \boxtimes \mathbb{1}, \quad H_\omega^r \stackrel{\text{def}}{=} \mathbb{1} \boxtimes H_\omega, \quad \mathbf{H}_\omega \stackrel{\text{def}}{=} H_\omega^l - H_\omega^r$$

Since  $\sum_{i=1}^N E_{ii} = \mathbb{1}$ , we may also write

$$H_\omega^l = -\frac{1}{\beta} \sum_{i,j=1}^N (\ln \lambda_i) \mathbf{P}_{ij}, \quad H_\omega^r = -\frac{1}{\beta} \sum_{i,j=1}^N (\ln \lambda_j) \mathbf{P}_{ij}.$$



Thus

$$\mathbf{H}_\omega = -\frac{1}{\beta} \sum_{i,j=1}^N \left( \ln \frac{\lambda_i}{\lambda_j} \right) \mathbf{P}_{ij}.$$

Using the operator:

$$\Delta_\omega \stackrel{\text{def}}{=} \sum_{i,j=1}^N \left( \frac{\lambda_i}{\lambda_j} \right) \mathbf{P}_{ij} = e^{-\beta \mathbf{H}_\omega},$$

we define a time evolution operator on  $\mathcal{C}_2$ :

$$e^{\mathbf{iH}_\omega t} = \Delta_\omega^{-\frac{it}{\beta}} \quad (t \in \mathbb{R}),$$

and we note that, for any  $X \in \mathcal{C}_2$ ,

$$\begin{aligned} e^{\mathbf{iH}_\omega t}(X) &= \sum_{i,j=1}^N \left( \frac{\lambda_i}{\lambda_j} \right)^{-\frac{it}{\beta}} \mathbf{P}_{ij}(X) \\ &= \left[ \sum_{i=1}^N \lambda_i^{-\frac{it}{\beta}} E_{ii} \right] \boxtimes \left[ \sum_{j=1}^N \lambda_j^{-\frac{it}{\beta}} E_{jj} \right] (X) \\ &= e^{\mathbf{iH}_\omega t} X e^{-\mathbf{iH}_\omega t}, \end{aligned}$$

so that

$$e^{\mathbf{iH}_\omega t} = e^{\mathbf{iH}_\omega t} \boxtimes e^{\mathbf{iH}_\omega t}.$$

It is clearly that  $\Omega$  commutes with  $H_\omega$  and hence that it is invariant under this time evolution:

$$e^{\mathbf{iH}_\omega t}(\Omega) = e^{\mathbf{iH}_\omega t} \Omega e^{-\mathbf{iH}_\omega t} = \Omega.$$

Finally, using  $e^{\mathbf{iH}_\omega t}(\Omega)$  we define the time evolution  $\sigma^\omega$  on the algebra  $\mathcal{A}_l$ , in the manner:

$$\sigma_t^\omega(A_l) = e^{\mathbf{iH}_\omega t} A_l e^{-\mathbf{iH}_\omega t}, \quad \forall A_l \in \mathcal{A}_l.$$

Writing  $A_l = A \boxtimes \mathbb{1}$ ,  $A \in \mathcal{L}(\mathcal{H})$ , and using the composition law, we see that

$$e^{\mathbf{iH}_\omega t} A_l e^{-\mathbf{iH}_\omega t} = \left[ e^{\mathbf{iH}_\omega t} A e^{-\mathbf{iH}_\omega t} \right] \boxtimes \mathbb{1},$$

so that

$$\begin{aligned} \omega(\sigma_t^\omega(A_l)) &= \text{Tr} \left( D_\omega e^{\mathbf{iH}_\omega t} A e^{-\mathbf{iH}_\omega t} \right) = \text{Tr} \left( e^{-\mathbf{iH}_\omega t} D_\omega e^{\mathbf{iH}_\omega t} A \right) \\ &= \text{Tr} (D_\omega A) = \omega(A_l), \end{aligned}$$

since  $D_\omega$  and  $H_\omega$  commute. Thus, the state  $\omega$  is invariant under the time evolution  $\sigma^\omega$ .

To obtain the KMS condition, we first note that, with  $A_l = A \boxtimes \mathbb{1}$  and  $B_l = B \boxtimes \mathbb{1}$ ,

$$A_l \sigma_t^\omega(B_l) = \left[ A e^{iH_\omega t} B e^{-iH_\omega t} \right] \boxtimes \mathbb{1}.$$

Hence,

$$\begin{aligned} F_{A_l, B_l}(t) &= \omega(A_l \sigma_t^\omega(B_l)) = \text{Tr} \left( D_\omega A e^{iH_\omega t} B e^{-iH_\omega t} \right) \\ &= \text{Tr} \left( e^{-iH_\omega t} D_\omega A e^{iH_\omega t} B \right) = \text{Tr} \left( D_\omega e^{-iH_\omega t} A e^{iH_\omega t} B \right), \end{aligned}$$

the last equality following from the commutativity of  $D_\omega$  and  $H_\omega$ . Thus, since  $D_\omega = e^{-\beta H_\omega}$ , that is,  $D_\omega e^{\beta H_\omega} = \mathbb{1}$ . Thus

$$\begin{aligned} F_{A_l, B_l}(t + i\beta) &= \text{Tr} \left( D_\omega e^{-iH_\omega t} e^{\beta H_\omega} A e^{iH_\omega t} e^{-\beta H_\omega} B \right) \\ &= \text{Tr} \left( D_\omega e^{\beta H_\omega} e^{-iH_\omega t} A e^{iH_\omega t} e^{-\beta H_\omega} B \right) \\ &= \text{Tr} \left( e^{-iH_\omega t} A e^{iH_\omega t} D_\omega B \right) = \text{Tr} \left( e^{iH_\omega t} D_\omega B e^{-iH_\omega t} A \right) \\ &= \text{Tr} \left( D_\omega e^{iH_\omega t} B e^{-iH_\omega t} A \right), \end{aligned}$$

so that

$$\omega(A_l \sigma_{t+i\beta}^\omega(B_l)) = \text{Tr} \left( D_\omega e^{iH_\omega t} B e^{-iH_\omega t} A \right) = \omega(\sigma_t^\omega(B_l) A_l),$$

which is the KMS condition.

• **The anti-linear operator  $S_\omega$ .**

We now analyze the anti-linear operator  $S_\omega : \mathcal{C}_2 \longrightarrow \mathcal{C}_2$ , which acts as

$$S_\omega(A_l \Omega) = A_l^* \Omega, \quad \forall A_l \in \mathcal{A}_l.$$

Taking  $A_l = A \boxtimes \mathbb{1}$ ,

$$S_\omega(A_l \Omega) = A_l^* \Omega, \quad \forall A_l \in \mathcal{A}_l \iff S_\omega(A \Omega) = A^* \Omega, \quad \forall A \in \mathcal{L}(\mathcal{H}).$$

Moreover, we may write,

$$S_\omega(A \Omega) = A^* \Omega \implies \sum_{i=1}^N \sqrt{\lambda_i} S_\omega(A E_{ii}) = \sum_{i=1}^N \sqrt{\lambda_i} A^* E_{ii}.$$

Taking  $A = E_{kl}$  and using  $E_{kl} E_{ii} = \delta_{li} E_{ki}$ , we then get

$$\sqrt{\lambda_l} S_\omega(E_{kl}) = \sqrt{\lambda_k} E_{lk} \implies S_\omega(E_{kl}) = \sqrt{\frac{\lambda_k}{\lambda_l}} E_{lk}.$$

Since any  $A \in \mathcal{L}(\mathcal{H})$  can be written as  $A = \sum_{i,j=1}^N a_{ij} E_{ij}$ , where  $a_{ij} = \langle \psi_i | A | \psi_j \rangle$ , and furthermore, since  $\mathbf{P}_{ij}(E_{kl}) = \delta_{ik} \delta_{jl} E_{ij}$ , we obtain

$$S_\omega = J \Delta_\omega^{\frac{1}{2}},$$

which in fact, also gives the polar decomposition of  $S_\omega$ .

Thus, we could have obtained the time evolution automorphisms  $\sigma_t^\omega (t \in \mathbb{R})$ , by analyzing the anti-linear operator  $S_\omega$ , (since  $S_\omega^* S_\omega = \Delta_\omega$ ) directly. Also, we see that the modular operator simply defines the *Gibbs state* corresponding to the Hamiltonian  $\mathbf{H}_\omega$ .

• **The centralizer.**

The centralizer of  $\mathcal{A}_l$ , with respect to the state  $\omega$ , is the von Neumann algebra,

$$\mathcal{M}_\omega = \{B_l \in \mathcal{A}_l : \omega([B_l, A_l]) = 0, \forall A_l \in \mathcal{A}_l\}.$$

Let us determine this von Neumann algebra. Writing  $A_l = A \boxtimes \mathbb{1}$ ,  $B_l = B \boxtimes \mathbb{1}$ , the commutator,  $[B_l, A_l] = (AB - BA) \boxtimes \mathbb{1}$ . Hence

$$\omega([B_l, A_l]) = \text{Tr}(D_\omega(AB - BA)).$$

Thus, in order for the above expression to vanish, we must have,

$$\sum_{i=1}^N \lambda_i \langle \psi_i | AB | \psi_i \rangle = \sum_{i=1}^N \lambda_i \langle \psi_i | BA | \psi_i \rangle, \quad \forall A \in \mathcal{L}(\mathcal{H}).$$

Taking  $A = |\psi_k\rangle\langle\psi_l|$ , this gives,

$$\lambda_k \langle \psi_l | B | \psi_k \rangle = \lambda_l \langle \psi_l | B | \psi_k \rangle, \quad \forall k, l = 1, \dots, N,$$

and since in general,  $\lambda_k \neq \lambda_l$ , this implies that  $\langle \psi_l | B | \psi_k \rangle = 0$  whenever  $k \neq l$ . Thus,  $B$  is of the general form  $B = \sum_{i=1}^N b_i E_{ii}$ ,  $b_i \in \mathbb{C}$ . In other words, the centralizer  $\mathcal{M}_\omega$  is generated by the projectors  $E_{ii}^l = E_{ii} \boxtimes \mathbb{1}$ ,  $i = 1, \dots, N$ , which are *minimal* (i.e., they do not contain projectors onto smaller subspaces) in  $\mathcal{A}_l$ . Alternatively, we may write,  $\mathcal{M}_\omega = \{H_\omega^l\}''$ , where  $H_\omega^l$  is the Hamiltonian defined above, so that it is an *atomic, commutative* von Neumann algebra.

## 4 Araki relative modular theory

Consider a von Neumann algebra  $\mathcal{M}$  in its standard form. If  $\mathcal{M}$  has the standard form  $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ , then  $\mathcal{M}$  acts on the Hilbert space  $\mathcal{H}$ ,  $J$  is the modular conjugation, and  $\mathcal{P}$  is

a natural positive cone in  $\mathcal{H}$  such that every faithful normal state  $\omega$  has a unique vector representative  $|\Omega\rangle$  in  $\mathcal{P}$  which is cyclic and separating for  $\mathcal{M}$ . Given another normal state  $\phi$ , the *densely defined quadratic form*

$$A|\Omega\rangle \mapsto \phi(AA^*), \quad \forall A \in \mathcal{M} \quad (4.1)$$

is *closable* and there exists an associated positive self-adjoint operator  $\Delta$ . It is characterized by the following properties.  $\mathcal{M}|\Omega\rangle$  is a *core* for  $\Delta^{\frac{1}{2}}$  and

$$\left\| \Delta^{\frac{1}{2}} A |\Omega\rangle \right\|^2 = \phi(AA^*).$$

The  $\Delta$  was called by Araki the *relative modular operator*<sup>1</sup> of  $\phi$  and  $\omega$  and it is usually denoted by  $\Delta(\phi/\omega)$  or  $\Delta_{\phi,\omega}$ . Equivalently,  $\Delta_{\phi,\omega}$  is obtained from the polar decomposition of the *closure*  $S_{\phi,\omega}$  of the conjugate linear operator

$$A|\Omega\rangle \mapsto A^*|\Phi\rangle,$$

where  $|\Phi\rangle$  is the vector representative of  $\phi$  from  $\mathcal{P}$ . Namely,

$$S_{\phi,\omega} = J\Delta_{\phi,\omega}^{\frac{1}{2}}.$$

The operators  $J, \Delta_{\omega,\omega}$  and  $\sigma_t^\omega$  are the standard ingredients of the Tomita-Takesaki modular theory with respect to  $\omega$  or  $|\Omega\rangle$ . The modular group of  $\omega$  is a one-parameter group of automorphisms of  $\mathcal{M}$  and it looks like

$$\sigma_t^\omega(A) = \Delta_{\omega,\omega}^{it} A \Delta_{\omega,\omega}^{-it}. \quad (4.2)$$

Another *Radon-Nikodym derivative-like* object for comparison of two states is the *Radon-Nikodym cocycle* discovered by Connes<sup>2</sup>. If  $\phi$  is a faithful normal state, then

$$[D\phi, D\omega]_t \stackrel{\text{def}}{=} \Delta_{\phi,\omega}^{it} \Delta_{\omega,\omega}^{-it} \equiv U_t \quad (4.3)$$

is a  $\sigma_t^\omega$ -cocycle and

$$\sigma_t^\phi = U_t \sigma_t^\omega U_t^*. \quad (4.4)$$

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<sup>1</sup>H. Araki and T. Masuda: Positive cones and  $L_p$ -spaces for von Neumann algebras, Publ. RIMS, Kyoto Univ. **18**, 339-411 (1982).

<sup>2</sup>A. Connes: Une classification des facteurs de type III, Ann. Ec. Norm. Sup. **6**, 133-252 (1973).

## 4.1 Functional calculus for a class of super-operators

We introduce two linear super-operators on the space  $M_d(\mathbb{C})$  of  $d \times d$  matrices. Left multiplication by  $A$  is denoted by  $\mathbb{L}_A$  and defined as

$$\mathbb{L}_A(X) \stackrel{\text{def}}{=} AX;$$

right multiplication by  $B$  is denoted  $\mathbb{R}_B$  and defined as

$$\mathbb{R}_B(X) \stackrel{\text{def}}{=} XB.$$

These super-operators are associated with the relative modular operator

$$\Delta_{A,B} = \mathbb{L}_A \mathbb{R}_B^{-1}$$

introduced by Araki in a far more general context. They have the following properties:

- (i) The super-operators  $\mathbb{L}_A, \mathbb{R}_B$  commute, i.e.  $[\mathbb{L}_A, \mathbb{R}_B] = 0$  since

$$\mathbb{L}_A \mathbb{R}_B(X) = AXB = \mathbb{R}_B \mathbb{L}_A(X)$$

even when  $A$  and  $B$  do not commute, i.e.  $[A, B] \neq 0$ .

- (ii)  $\mathbb{L}_A$  and  $\mathbb{R}_A$  are invertible if and only if  $A$  is non-singular, in which case

$$\mathbb{L}_A^{-1} = \mathbb{L}_{A^{-1}} \quad \text{and} \quad \mathbb{R}_A^{-1} = \mathbb{R}_{A^{-1}}.$$

- (iii) When  $A$  is self-adjoint,  $\mathbb{L}_A$  and  $\mathbb{R}_A$  are both self-adjoint with respect to the Hilbert-Schmidt inner product  $\langle A, B \rangle_{\text{HS}} \stackrel{\text{def}}{=} \text{Tr}(A^* B)$ .

- (iv) When  $A \geq 0$ , the super-operators  $\mathbb{L}_A$  and  $\mathbb{R}_A$  are positive semi-definite, i.e.

$$\langle X, \mathbb{L}_A(X) \rangle_{\text{HS}} = \text{Tr}(X^* AX) \geq 0, \quad \langle X, \mathbb{R}_A(X) \rangle_{\text{HS}} = \text{Tr}(X^* XA) = \text{Tr}(XAX^*) \geq 0.$$

- (v) When  $A \geq 0$ , then

$$(\mathbb{L}_A)^\alpha = \mathbb{L}_{A^\alpha}, \quad (\mathbb{R}_A)^\alpha = \mathbb{R}_{A^\alpha}$$

for all  $\alpha \geq 0$ . If  $A > 0$ , this extends to all real  $\alpha$ . More generally,

$$f(\mathbb{L}_A) = \mathbb{L}_{f(A)}$$

for all  $f : (0, +\infty) \rightarrow (-\infty, +\infty)$ .

## 4.2 Version of super-operator representation

Suppose that  $\Omega$  and  $\Phi$  are *separating* and *cyclic* vectors, induced by faithful normal states  $\omega$  and  $\phi$ , respectively, in  $\mathcal{C}_2$  for  $\mathcal{A}_l$ . Then there exist two *non-singular* density operators  $D_\omega, D_\phi \in \mathcal{C}_2$  such that

$$\Omega = D_\omega^{\frac{1}{2}}, \quad \Phi = D_\phi^{\frac{1}{2}},$$

According the Araki relative modular theory, we have that for any  $X_l \in \mathcal{A}_l$  and  $Y_r \in \mathcal{A}_r$ ,

$$\begin{cases} S_{\phi, \omega}(X_l \Omega) &= X_l^* \Phi, \\ F_{\phi, \omega}(Y_r \Omega) &= Y_r^* \Phi. \end{cases} \quad (4.5)$$

Both expressions are equivalent to

$$\begin{cases} S_{\phi, \omega} \left( X D_\omega^{\frac{1}{2}} \right) &= X^* D_\phi^{\frac{1}{2}}, \\ F_{\phi, \omega} \left( D_\omega^{\frac{1}{2}} Y \right) &= D_\phi^{\frac{1}{2}} Y^*, \end{cases} \quad (4.6)$$

for any  $X, Y \in L(\mathcal{H})$ . Thus if the dimension of the underlying Hilbert space  $\mathcal{H}$  satisfies that  $\dim(\mathcal{H}) < +\infty$ , then

$$\begin{cases} S_{\phi, \omega}(A) &= D_\omega^{-\frac{1}{2}} A^* D_\phi^{\frac{1}{2}}, \\ F_{\phi, \omega}(B) &= D_\phi^{\frac{1}{2}} B^* D_\omega^{-\frac{1}{2}}, \end{cases} \quad (4.7)$$

for any  $A, B \in \mathcal{C}_2$ .

$$\Delta_{\phi, \omega} = F_{\phi, \omega} S_{\phi, \omega} = D_\phi \boxtimes D_\omega^{-1},$$

which implies that

$$\begin{cases} J_{\phi, \omega} X &= X^*, \\ \Delta_{\phi, \omega}^{\frac{1}{2}} \Omega &= \Phi, \\ \Delta_{\phi, \omega}^{it} &= D_\phi^{it} \boxtimes D_\omega^{-it}. \end{cases} \quad (4.8)$$

## 4.3 Version of vector representation

Suppose that  $|\Omega\rangle$  and  $|\Phi\rangle$  are separating and cyclic vectors, induced by faithful normal states  $\omega$  and  $\phi$ , respectively, in  $\mathcal{H} \equiv \mathcal{H}_d \otimes \mathcal{H}_d$  for  $\pi(\mathcal{M}) \equiv \mathcal{M} \otimes \mathbb{1}_d$  with  $\mathcal{M} = L(\mathcal{H}_d)$ . Then there exist two non-singular density operators  $D_\omega, D_\phi \in L(\mathcal{H}_d)$  such that their

purifications are  $|\Omega\rangle = \text{vec}(D_\omega^{\frac{1}{2}})$  and  $|\Phi\rangle = \text{vec}(D_\phi^{\frac{1}{2}})$ . According the Araki relative modular theory, we have that for any  $X, Y \in L(\mathcal{H}_d)$ ,

$$\begin{cases} S_{\phi,\omega}(X \otimes \mathbb{1}_d)|\Omega\rangle &= (X^* \otimes \mathbb{1}_d)|\Phi\rangle, \\ F_{\phi,\omega}(\mathbb{1}_d \otimes Y)|\Omega\rangle &= (\mathbb{1}_d \otimes Y^*)|\Phi\rangle. \end{cases} \quad (4.9)$$

Both expressions are equivalent to

$$\begin{cases} S_{\phi,\omega} \text{vec}(XD_\omega^{\frac{1}{2}}) &= \text{vec}(X^*D_\phi^{\frac{1}{2}}), \\ F_{\phi,\omega} \text{vec}(D_\omega^{\frac{1}{2}}Y) &= \text{vec}(D_\phi^{\frac{1}{2}}Y^*), \end{cases} \quad (4.10)$$

for any  $X, Y \in L(\mathcal{H}_d)$ . Thus

$$\begin{cases} S_{\phi,\omega} \text{vec}(X) &= \text{vec}(D_\omega^{-\frac{1}{2}}X^*D_\phi^{\frac{1}{2}}), \\ F_{\phi,\omega} \text{vec}(Y) &= \text{vec}(D_\phi^{\frac{1}{2}}Y^*D_\omega^{-\frac{1}{2}}), \end{cases} \quad (4.11)$$

for any  $X, Y \in L(\mathcal{H}_d)$ .

$$\Delta_{\phi,\omega} = FS = D_\phi \otimes (D_\omega^{-1})^\top,$$

which implies that

$$\begin{cases} J_{\phi,\omega} \text{vec}(X) &= \text{vec}(X^*), \\ \Delta_{\phi,\omega}^{\frac{1}{2}}|\Omega\rangle &= |\Phi\rangle, \\ \Delta_{\phi,\omega}^{it} &= D_\phi^{it} \otimes (D_\omega^{-it})^\top. \end{cases} \quad (4.12)$$

## 5 Specific form of natural positive cone

Let  $\mathcal{H}_d$  be a  $d$ -dimensional complex Hilbert space. Consider a von Neumann algebra  $\mathcal{M} \equiv L(\mathcal{H}_d)$ . A faithful representation of von Neumann algebra  $\mathcal{M}$  on a Hilbert space  $\mathcal{H} \equiv \mathcal{H}_d \otimes \mathcal{H}_d$  is defined by the following map:

$$\pi : X \longmapsto \pi(X) = X \otimes \mathbb{1}_d.$$

$|\Omega\rangle \stackrel{\text{def}}{=} \text{vec}(\mathbb{1}_d)$  is a separating and cyclic vector in  $\mathcal{H}$  for von Neumann algebra  $\pi(\mathcal{M}) \equiv L(\mathcal{H}_d) \otimes \mathbb{1}_d$ . Thus von Neumann algebra  $\mathcal{M}$  have a standard representation  $(\pi(\mathcal{M}), \mathcal{H}, |\Omega\rangle)$ .

According to the definition of the natural positive cone  $\mathcal{P}$  associated with the pair  $(\pi(\mathcal{M}), |\Omega\rangle)$  is the closure of the set:

$$\{\pi(M)j(\pi(M))|\Omega\rangle : M \in \mathcal{M}\},$$

where  $j : \pi(\mathcal{M}) \mapsto \pi(\mathcal{M})'$  is the anti-linear  $*$ -isomorphism defined by

$$j(\pi(M)) \stackrel{\text{def}}{=} J\pi(M)J, \quad \forall M \in \mathcal{M}.$$

More concretely,

$$\begin{aligned} \pi(M)j(\pi(M))|\Omega\rangle &= (M \otimes \mathbb{1}_d)J(M \otimes \mathbb{1}_d)J \text{vec}(\mathbb{1}_d) \\ &= (M \otimes \mathbb{1}_d)J(M \otimes \mathbb{1}_d) \text{vec}(\mathbb{1}_d) \\ &= (M \otimes \mathbb{1}_d)J \text{vec}(M) = (M \otimes \mathbb{1}_d) \text{vec}(M^*) \\ &= \text{vec}(MM^*), \end{aligned}$$

which indicate that

$$\mathcal{P} = [\{\text{vec}(MM^*) : M \in \mathcal{M}\}] = [\text{vec}(\mathcal{M}^+)] = \text{vec}(\mathcal{M}^+).$$

For any  $|\xi\rangle \in \mathcal{P}$ , there exists an element  $X \in \mathcal{M}^+$  such that  $|\xi\rangle = \text{vec}(X)$ , thus  $J \text{vec}(X) = \text{vec}(X^*) = \text{vec}(X)$  since  $X = X^*$ . Therefore  $J|\xi\rangle = |\xi\rangle$ . For any  $\text{vec}(NN^*) \in \mathcal{P}$  for some  $N \in \mathcal{M}$ , we have

$$\pi(M)j(\pi(M)) \text{vec}(NN^*) = \text{vec}(MNN^*M^*) = \text{vec}((MN)(MN)^*) \in \mathcal{P}.$$

$|\xi\rangle, |\eta\rangle$  are any given vectors in  $\mathcal{P}$ . There exist two elements  $X, Y \in \mathcal{M}^+$  such that  $|\xi\rangle = \text{vec}(X)$  and  $|\eta\rangle = \text{vec}(Y)$ . Then

$$\langle \xi | \eta \rangle = \langle \text{vec}(X), \text{vec}(Y) \rangle = \langle X, Y \rangle_{\text{HS}} = \text{Tr}(XY) \geq 0$$

since  $X, Y \geq 0$ . Thus  $\mathcal{P}$  is a self-dual cone. If  $Z \in \mathcal{M}^+$  such that  $\text{vec}(Z) \in \mathcal{P} \cap (-\mathcal{P})$ , then  $\text{vec}(Z) \in \mathcal{P}$  and  $\text{vec}(-Z) \in \mathcal{P}$ , which implies that  $-Z, Z \geq 0$ , i.e.  $Z = 0 \iff \text{vec}(Z) = 0$ . Therefore  $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$ .

If  $|\zeta\rangle$  satisfies that  $J|\zeta\rangle = |\zeta\rangle$ , then there is an element  $T \in \mathcal{M}$  such that  $|\zeta\rangle = \text{vec}(T)$  and  $\text{vec}(T) = J \text{vec}(T)$ , which is equivalent to the following formula:

$$\text{vec}(T) = \text{vec}(T^*) \iff T = T^*.$$

Now by employing the Jordan decomposition of operators, we have

$$T = T^+ - T^-,$$



where  $T^+, T^- \in \mathcal{M}^+$  and  $T^+T^- = 0$ . This means that

$$|\zeta\rangle = \text{vec}(T) = \text{vec}(T^+) - \text{vec}(T^-)$$

and  $\langle \text{vec}(T^+), \text{vec}(T^-) \rangle = \langle T^+, T^- \rangle_{\text{HS}} = \text{Tr}(T^+T^-) = 0$ . Denote  $|\zeta_1\rangle = \text{vec}(T^+)$  and  $|\zeta_2\rangle = \text{vec}(T^-)$ , then  $|\zeta\rangle = |\zeta_1\rangle - |\zeta_2\rangle$  with  $|\zeta_1\rangle \perp |\zeta_2\rangle$ .

For any  $|\zeta\rangle \in \mathcal{H}$ , there is an element  $Y_{|\zeta\rangle} \in \mathcal{M}$  such that  $|\zeta\rangle = \text{vec}(Y_{|\zeta\rangle})$ . Now since  $Y_{|\zeta\rangle}$  can be represented by at most four positive element in  $H^+, H^-, K^+, K^- \in \mathcal{M}^+$  as follows:

$$Y_{|\zeta\rangle} = (H^+ - H^-) + i(K^+ - K^-),$$

i.e.,

$$\text{vec}(Y_{|\zeta\rangle}) = \text{vec}(H^+) - \text{vec}(H^-) + i \text{vec}(K^+) - \text{vec}(K^-).$$

Setting  $\text{vec}(H^+) = |\zeta_1\rangle, \text{vec}(H^-) = |\zeta_2\rangle, \text{vec}(K^+) = |\zeta_3\rangle$  and  $\text{vec}(K^-) = |\zeta_4\rangle$ , we have

$$|\zeta\rangle = |\zeta_1\rangle - |\zeta_2\rangle + i|\zeta_3\rangle - i|\zeta_4\rangle.$$

Clearly,  $|\zeta_1\rangle, |\zeta_2\rangle, |\zeta_3\rangle, |\zeta_4\rangle \in \mathcal{P}$ . Finally,  $\mathcal{H}$  indeed is linearly spanned by  $\mathcal{P}$ .

Since any normal positive form  $\omega \in \mathcal{M}_{*,+}$ , it follows that  $|\Omega\rangle = \text{vec}(D_\omega^{\frac{1}{2}})$  is the vector representative of  $|\omega\rangle$  in  $\mathcal{P}$ :  $\omega(M) = \langle \Omega | \pi(M) | \Omega \rangle$ .

Given any normal positive forms  $\omega_{|\xi\rangle}$  and  $\omega_{|\eta\rangle}$  for  $|\xi\rangle, |\eta\rangle \in \mathcal{P}$ , thus we have  $|\xi\rangle = \text{vec}(X)$  and  $|\eta\rangle = \text{vec}(Y)$  for  $X, Y \in \mathcal{M}^+$ :

$$\begin{aligned} \||\xi\rangle - |\eta\rangle\|^2 &= \langle \xi - \eta | \xi - \eta \rangle = \langle \text{vec}(X - Y), \text{vec}(X - Y) \rangle_{\text{HS}} \\ &= \|X - Y\|_{\text{HS}}^2, \end{aligned} \quad (5.1)$$

$$\||\xi\rangle - |\eta\rangle\| \||\xi\rangle + |\eta\rangle\| = \|X - Y\|_{\text{HS}} \|X + Y\|_{\text{HS}}, \quad (5.2)$$

$$\|\omega_{|\xi\rangle} - \omega_{|\eta\rangle}\| = \|X^2 - Y^2\|_1. \quad (5.3)$$

In what follows, we prove the following inequality:

**Theorem 5.1.**

$$\|X - Y\|_{\text{HS}}^2 \leq \|X^2 - Y^2\|_1 \leq \|X - Y\|_{\text{HS}} \|X + Y\|_{\text{HS}}. \quad (5.4)$$

*Proof.* Since

$$X^2 - Y^2 = \frac{1}{2} [(X - Y)(X + Y) + (X + Y)(X - Y)],$$

it follows that

$$\left\| X^2 - Y^2 \right\|_1 \leq \frac{1}{2} \left\| (X - Y)(X + Y) \right\|_1 + \frac{1}{2} \left\| (X + Y)(X - Y) \right\|_1.$$

By employing Schwarz inequality, we have

$$\left\{ \begin{array}{l} \left\| (X - Y)(X + Y) \right\|_1 \\ \left\| (X + Y)(X - Y) \right\|_1 \end{array} \right\} \leq \|X - Y\|_{\mathbf{HS}} \|X + Y\|_{\mathbf{HS}}.$$

Thus

$$\left\| X^2 - Y^2 \right\|_1 \leq \|X - Y\|_{\mathbf{HS}} \|X + Y\|_{\mathbf{HS}}.$$

Next, we write the spectral decomposition of  $X - Y$  as follows:

$$X - Y = \sum_i \lambda_i |u_i\rangle \langle u_i|.$$

Then

$$|X - Y| = \sum_i |\lambda_i| |u_i\rangle \langle u_i|, \quad \langle u_i | X - Y | u_i \rangle = \lambda_i.$$

Denote

$$U \stackrel{\text{def}}{=} \sum_i \text{sign}(\lambda_i) |u_i\rangle \langle u_i|.$$

Thus  $[U, X - Y] = 0$  and  $|X - Y| = U(X - Y) = (X - Y)U$ . Now by the triangle inequality, we have

$$\begin{aligned} |\lambda_i| &= |\langle u_i | X - Y | u_i \rangle| = |\langle u_i | X | u_i \rangle - \langle u_i | Y | u_i \rangle| \\ &\leq \langle u_i | X | u_i \rangle + \langle u_i | Y | u_i \rangle \\ &\leq \langle u_i | X + Y | u_i \rangle. \end{aligned} \tag{5.5}$$

Therefore

$$\begin{aligned} \left\| X^2 - Y^2 \right\|_1 &\geq \left| \text{Tr} \left( [X^2 - Y^2] U \right) \right| \\ &= \left| \frac{1}{2} \text{Tr} ((X - Y)(X + Y)U) + \frac{1}{2} \text{Tr} ((X + Y)(X - Y)U) \right| \\ &= \frac{1}{2} |\text{Tr} (|X - Y| (X + Y)) + \text{Tr} ((X + Y) |X - Y|)| \\ &= \text{Tr} (|X - Y| (X + Y)) = \sum_i |\lambda_i| \text{Tr} (|u_i\rangle \langle u_i| (X + Y)) \\ &= \sum_i |\lambda_i| \langle u_i | X + Y | u_i \rangle \geq \sum_i |\lambda_i|^2 = \|X - Y\|_{\mathbf{HS}}^2. \end{aligned}$$

The desired inequality is obtained. □

Powers-Störmer's inequality asserts that for  $s \in [0, 1]$ , the following inequality

$$2 \operatorname{Tr} \left( A^s B^{1-s} \right) \geq \operatorname{Tr} (A + B - |A - B|) \quad (5.6)$$

holds for any pair of positive matrices  $A, B$ . This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory<sup>3</sup>. This inequality was first proven by Audenaert, using an integral representation of the function  $t^s$ . After that, Ozawa gave a much simpler proof for the same inequality, using fact<sup>4</sup> that  $f(t) = t^s, t \in [0, +\infty)$  is an operator monotone function for  $s \in [0, 1]$ .

**Theorem 5.2** (Powers-Störmer inequality<sup>5</sup>, 1970). *For positive compact operators  $A, B$ , the following inequality is valid:*

$$\left\| \sqrt{A} - \sqrt{B} \right\|_2^2 \leq \|A - B\|_1.$$

**Theorem 5.3.** *Let  $A, B$  be semi-definite positive matrices in  $M_n(\mathbb{C})$ . Then*

$$2 \operatorname{Tr} \left( B^s A^{1-s} \right) \geq \operatorname{Tr} (A + B - |A - B|)$$

*holds for any  $s \in [0, 1]$ .*

*Proof.* (N. Ozawa, unpublished) For  $X$  self-adjoint,  $X_{\pm}$  denotes its positive/negative part. Decomposing  $A - B = (A - B)_{+} - (A - B)_{-}$ , one gets

$$\frac{1}{2} \operatorname{Tr} (A + B - |A - B|) = \operatorname{Tr} (A) - \operatorname{Tr} ((A - B)_{+}).$$

Now the original inequality is equivalent to

$$\operatorname{Tr} (A) - \operatorname{Tr} \left( B^s A^{1-s} \right) \leq \operatorname{Tr} ((A - B)_{+}). \quad (5.7)$$

Note that

$$B + (A - B)_{+} \geq B \quad \text{and} \quad B + (A - B)_{+} = A + (A - B)_{-} \geq A.$$

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<sup>3</sup>K.M.R. Audenaert et al. The quantum Chernoff bound, Phys. Rev. Lett. **98**, 16050 (2007).

<sup>4</sup>V. Jakšić et al. Quantum hypothesis testing and non-equilibrium statistical mechanics. Rev. Math. Phys. **24**, 1230002 (2012).

<sup>5</sup>R.T. Powers: Commun. Math. Phys. **16**, 1-33 (1970).

Since, for  $s \in [0, 1]$ , the function  $x \mapsto x^s$  is operator monotone, i.e.  $X \leq Y \implies X^s \leq Y^s$  for any positive matrices  $X, Y$ , we can write

$$\begin{aligned}
\operatorname{Tr}(A) - \operatorname{Tr}(B^s A^{1-s}) &= \operatorname{Tr}((A^s - B^s)A^{1-s}) \\
&\leq \operatorname{Tr}(((B + (A - B)_+)^s - B^s)A^{1-s}) \\
&\leq \operatorname{Tr}(((B + (A - B)_+)^s - B^s)(B + (A - B)_+)^{1-s}) \\
&= \operatorname{Tr}(B + (A - B)_+) - \operatorname{Tr}(B^s(B + (A - B)_+)^{1-s}) \\
&\leq \operatorname{Tr}(B + (A - B)_+).
\end{aligned}$$

□

**Theorem 5.4** (Ogata<sup>6</sup>, 2011). *Let  $\phi_1, \phi_2$  are normal positive linear functionals on a von Neumann algebra  $\mathcal{M}$  for which the vector representatives in the natural positive cone  $\mathcal{P}$  are  $|\Phi_1\rangle$  and  $|\Phi_2\rangle$ , respectively. Then we have that,  $\forall s \in [0, 1]$ ,*

$$2 \left\| \Delta_{\phi_2, \phi_1}^{\frac{s}{2}} |\Phi_1\rangle \right\|^2 \geq \phi_1(\mathbb{1}) + \phi_2(\mathbb{1}) - |\phi_1 - \phi_2|(\mathbb{1}). \quad (5.8)$$

*The equality holds if and only if*

$$\phi_2 = (\phi_2 - \phi_1)_+ + \psi \quad \text{and} \quad \phi_1 = (\phi_2 - \phi_1)_- + \psi$$

*for some normal positive linear functional  $\psi$  on  $\mathcal{M}$  whose support is orthogonal to the support of  $|\phi_2 - \phi_1|$ .*

**Theorem 5.5** (D.T. Hoa<sup>7</sup>, 2012). *Let  $f$  be a  $2n$ -monotone function on  $[0, +\infty)$  such that  $f((0, +\infty)) \subseteq (0, +\infty)$ . Then for any pair of positive matrices  $A, B \in M_n(\mathbb{C})$ , we have:*

$$2 \operatorname{Tr} \left( \sqrt{f(A)} g(B) \sqrt{f(A)} \right) \geq \operatorname{Tr}(A + B - |A - B|), \quad (5.9)$$

*where*

$$g(t) \stackrel{\text{def}}{=} \begin{cases} \frac{t}{f(t)}, & t \in (0, +\infty), \\ 0, & t = 0. \end{cases}$$

<sup>6</sup>Y. Ogata: A Generalization of Powers-Störmer Inequality. Lett Math Phys **97**, 339-346 (2011).

<sup>7</sup>D.T. Hoa et al. On generalized Powers-Störmer's inequality. Linear Algebra and Its Applications 438, 242-249 (2012).

**Theorem 5.6** (D.T. Hoa, 2012). *Let  $\tau$  be a tracial functional on a  $C^*$ -algebra  $\mathcal{A}$ ,  $f$  be a strictly positive, operator monotone function on  $[0, +\infty)$ . Then for any pair of positive elements  $A, B \in \mathcal{A}$ :*

$$2\tau \left( \sqrt{f(A)}g(B)\sqrt{f(A)} \right) \geq \tau(A + B - |A - B|), g(t) \stackrel{\text{def}}{=} t/f(t). \quad (5.10)$$

**Theorem 5.7** (J. Phillips<sup>8</sup>, 1986). *Let  $A \geq B \geq 0$  and  $t \geq 1$ . Then*

$$\left\| A^{1/t} - B^{1/t} \right\|_t^t \leq \|A - B\|_1.$$

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<sup>8</sup>J. Phillips: Generalized Powers-Størmer inequalities. Talk at the Canadian Operator Theory Conference, Victoria, July 1986 (unpublished).